Explicit Travelling Wave Solutions to Nonlinear Partial Differential Equations Arise in Mathematical Physics and Engineering

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ABSTRACT

To describe the interior phenomena of the mysterious problems around the real world, non-linear partial differential equations (NLPDEs) play a substantial role, for which construction of analytic solutions of those is most important. This paper stands for a goal to find fresh and wide-ranging solutions to some familiar NLPDEs namely the non-linear cubic Klein-Gordon (cKG) equation and the non-linear Benjamin-Ono (BO) equation. A wave variable transformation is made use to convert the mentioned equations into ordinary differential equations. To acquire the desired precise exact travelling wave solutions to the above-stated equations, the rational \((G'/G)\)-expansion method is employed. Consequently, three types of equipped solutions are successfully come out in the forms of hyperbolic, trigonometric and rational functions in a compatible way. To analyse the physical problems arisen relating to nonlinear complex dynamical systems, our obtained solutions might be most helpful. So far we know, these achieved solutions are different than those in the literature. The applied method is efficient and reliable which might further be used to find different and novel solutions to many other NLPDEs successfully in research field.

Keywords: The rational \((G'/G)\)-expansion method, nonlinear partial differential equation, complex transformation, exact solution.

1 Introduction

In this new era, nonlinear phenomena have arisen in a wide range in the area of extensive physical science and mathematics. The nonlinear mechanism of nature can be depicted by NLPDEs. For this reason, with the rapid development of nonlinear sciences, it has debuted with a lot of importance in physical science and mathematics and many research works have been done to analyse these equations. A special case, the closed form solutions of NLPDEs bears significant role to delineate many models concerning the underlying mechanisms of real world. Subsequently, research on this topic is becoming as a matter of attraction in the field of nonlinear science day by day. With this importance, for the analytical solutions of NLPDEs, many traditional techniques have been emerged and implemented to solve various kinds of problems such as the Adomian decomposition method [1]-[2], the tanh function method [3]-[4], the simplest equation method [5]-[6], the Jacobi elliptic function method [7]-[8], the expansion function method [9]-[10], the modified simple equation method [11]-[12], the \((G'/G)\)-expansion method [13]-[14], the cosh function method [15]-[16], the homotopy perturbation method [17]-[18], the multiple exp-function method [19]-[20], the Bernoulli sub-ODE method [21]-[22], the homotopy analysis method [23]-[24], the variational iteration method [25]-[26], the modified tanh-coth method and the extended Jacobi elliptic function method [27], the He homotopy perturbation method [28]-[29], the homogeneous balance method [30], the inverse scattering method [31], the Backlund transformation method [32]-[33], the extended modified direct algebraic method [34]-[37]. In this paper, we have described the rational \((G'/G)\)-expansion method [38]-[41].

2 Explanation of the Technique

Consider the following NLPDE:

\[ F(u, u_x, u_t, u_{xx}, u_{tt}, u_{tx}, \ldots) = 0 \quad \text{(1)} \]

where \( u = u(x,t) \) and the subscripts in \( u \) represents partial derivatives. Followings are the main steps of the rational \((G'/G)\)-expansion method:

**First step:** Introduce the transformation

\[ u(x,t) = U(\xi), \quad \xi = x \pm vt \quad \text{(2)} \]

where \( v \) stands for the wave velocity. This transformation reduces Eq. (1) to the ordinary differential equation with respect to \( \xi \),

\[ Q(U, U', U'', U''', \ldots) = 0 \quad \text{(3)} \]

**Second step:** Take anti-derivative of Eq. (3) as much as possible; the integral constant may be considered as zero for seeking solitary wave solutions.

**Third step:** Consider the solution of Eq. (3) as follows:

\[ U(\xi) = \frac{\sum_{i=0}^{n} a_i (G'/G)^i}{\sum_{i=0}^{m} b_i (G'/G)^i} \quad \text{(4)} \]

with unknown constants \( a_i \) and \( b_i (i = 0,1,2,\ldots,n) \) in which at least one of \( a_n \) and \( b_n \) is non-zero. \( G = G(\xi) \) satisfies the ordinary differential equation,

\[ G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \quad \text{(5)} \]

where \( \lambda \) and \( \mu \) are real parameters. Eq. (5) has turned into

\[ \frac{d}{d\xi} \left( G'/G \right) = -(G'/G)^2 - \lambda(G'/G) - \mu \quad \text{(6)} \]

Then we have the general solutions of Eq. (5) (or equivalent to Eq. (6)) as follows:
\begin{equation}
(G'/G) = -\frac{1}{2} + \frac{\sqrt{2\mu} - 4\mu}{\sqrt{2\mu} - 4}\left(\frac{\text{Asinh}\left(\sqrt{\frac{2\mu}{4\mu - 2\mu/2}}\right) + B\cosh\left(\sqrt{\frac{2\mu}{4\mu - 2\mu/2}}\right)}{\text{Acosh}\left(\sqrt{\frac{2\mu}{4\mu - 2\mu/2}}\right) + B\sinh\left(\sqrt{\frac{2\mu}{4\mu - 2\mu/2}}\right)}\right), \quad \lambda^2 = 4\mu > 0
\end{equation}

**Fifth step:** Eq. (3) with Eqs. (4), (5) makes a polynomial \((G'/G)\). Set each coefficient to zero and solve them by the computer software Maple to calculate the values of \(a_1, b, \mu\) and \(r\).

**Sixth step:** Inserting the values determined in fifth step along with the outcomes given in Eqs. (7)-(9) into solution Eq. (4) provides the solutions of Eq. (1).

### 3 Implementation of the Technique

Herein, the suggested scheme is applied to examine the exact analytic solutions to the considered equations.

#### 3.1 The cKG equation

The cKG equation is

\begin{equation}
u_{xx} + u_{yy} - u_{tt} + au + bu^3 = 0
\end{equation}

The wave variable transformation \(u(x, y, t) = U(\xi)\), \(\xi = x + y - ct\) reduces Eq. (10) into the following equation:

\begin{equation}(2 - c^2)U'' + aU + bU^3 = 0\end{equation}

Due to homogeneous balance method, Eq. (11) gives \(n = 1\) and Eq. (4) turns into the form

\begin{equation}U(\xi) = \frac{a_0 + a_1 (G'/G)}{b_0 + b_1 (G'/G)}\end{equation}

Inserting Eq. (12) into Eq. (11) provides a polynomial in \((G'/G)\). Set the coefficients to zero and solved by computational software Maple. Accordingly, the following outcomes for \(a_0, a_1, b_0, b_1\) and \(c\) are obtained:

**Set-1:**

\begin{equation}a_0 = \pm\frac{b_1}{2} \sqrt{\frac{a(2\mu - 4\mu^2)}{b}}, \quad a_1 = 0, \quad b_0 = \frac{b_1 \lambda}{2}, \quad c = \pm\sqrt{\frac{2(a + 4\mu - \lambda^2)}{\lambda^2 + 4\mu}}\end{equation}

**Set-2:**

\begin{equation}a_0 = \mp(2b_1 \mu - \lambda b_0) \sqrt{\frac{a}{b(-\lambda^2 + 4\mu)}}, \quad a_1 = \pm(2b_0 - b_1 \lambda) \sqrt{\frac{a}{b(-\lambda^2 + 4\mu)}}, \quad c = \pm\frac{2(a + 4\mu - \lambda^2)}{(-\lambda^2 + 4\mu)}\end{equation}

**Set-3:**

\begin{equation}a_0 = \frac{b_1 (2\mu + \lambda \sqrt{12\mu + 3\lambda^2})}{b \sqrt{-12\mu + 3\lambda^2}}, \quad a_1 = \pm\frac{\sqrt{-3ab}}{3b} b_1, \quad b_0 = \frac{\lambda}{2} \pm \frac{\sqrt{-12\mu + 3\lambda^2}}{b \sqrt{-12\mu + 3\lambda^2}} b_1, \quad c = \pm\frac{2(a + 4\mu - \lambda^2)}{(-\lambda^2 + 4\mu)}\end{equation}

**Set-4:**

\begin{equation}a_0 = -\frac{b_1 (2\mu + \lambda \sqrt{12\mu + 3\lambda^2})}{b \sqrt{-12\mu + 3\lambda^2}}, \quad a_1 = \pm\frac{\sqrt{-3ab}}{3b} b_1, \quad b_0 = \frac{\lambda}{2} \pm \frac{\sqrt{-12\mu + 3\lambda^2}}{b \sqrt{-12\mu + 3\lambda^2}} b_1, \quad c = \pm\frac{2(a + 4\mu - \lambda^2)}{(-\lambda^2 + 4\mu)}\end{equation}

Eq. (12) along with Eq. (13) becomes

\begin{equation}U_1(\xi) = \sqrt[\mu]{\frac{a}{b \lambda + 2 (G'/G)}}, \end{equation}

Where,

\begin{equation}\xi = x + y \pm \sqrt[\mu]{\frac{2(a + 4\mu - \lambda^2)}{-\lambda^2 + 4\mu}} t\end{equation}

Utilizing the solutions in Eqs. (7)-(9) from Eq. (17), we make available the following solutions to Eq. (10) in the following three different forms:

**Case 1:** When \(\lambda^2 + 4\mu > 0\), Eq. (12) along with Eq. (13) becomes

\begin{equation}U_1(\xi) = \sqrt[\mu]{\frac{a}{b \lambda + 2 (G'/G)}}, \quad U_1(\xi) = \frac{a \text{ Acosh} \left(\sqrt{\frac{\lambda^2 + 4\mu}{2}}\right)}{b \lambda + 2 (G'/G)} + \frac{b \text{ sinh} \left(\sqrt{\frac{\lambda^2 + 4\mu}{2}}\right)}{b \lambda + 2 (G'/G)}\end{equation}

where,

\begin{equation}\xi = x + y \pm \sqrt[\mu]{\frac{2(a + 4\mu - \lambda^2)}{-\lambda^2 + 4\mu}} t\end{equation}
For $A = 0$ and $B \neq 0$, we get

$$U_{12}(\xi) = i \sqrt{\frac{\pi}{b}} \tanh \left( \sqrt{\lambda^2 - 4\mu/2} \xi \right)$$

(19)

Assigning $A \neq 0$ and $B = 0$ yields

$$U_{13}(\xi) = i \sqrt{\frac{\pi}{b}} \coth \left( \sqrt{\lambda^2 - 4\mu/2} \xi \right)$$

(20)

Case 2: For $\lambda^2 - 4\mu < 0$,

$$U_{14}(\xi) = \frac{\sqrt{\pi}}{2\mu} \left[ A \cos \left( \sqrt{\mu - \lambda^2/2} \xi \right) + B \sin \left( \sqrt{\mu - \lambda^2/2} \xi \right) \right]$$

(21)

where,

$$\xi = x + y \pm \frac{2(a+4\mu-\lambda^2)}{-2\mu^2+4\mu}\cdot t.$$

Applying $A = 0$ and $B \neq 0$, provides

$$U_{15}(\xi) = \frac{\sqrt{\pi}}{2\mu} \tan \left( \sqrt{4\mu - \lambda^2/2} \xi \right)$$

(22)

Using $A \neq 0$ and $B = 0$ gives

$$U_{16}(\xi) = -\sqrt{\frac{\pi}{b}} \cot \left( \sqrt{4\mu - \lambda^2/2} \xi \right)$$

(23)

Case 3: If $\lambda^2 - 4\mu = 0$, the method yields stationary wave solutions and thus have not been documented. Using the similar procedure for the all-other sets of solutions, as we have applied for set-1, we obtain the hyperbolic solutions for $\lambda^2 - 4\mu > 0$, the trigonometric solutions for $\lambda^2 - 4\mu < 0$, and $\lambda^2 - 4\mu = 0$ gives the stationary wave solutions.

3.2 The BO equation

Consider the BO equation

$$u_t + hu_{xx} + uu_x = 0$$

(24)

Using the transformation $u(x, t) = U(\xi)$, $\xi = x - ct$, Eq. (24) reduces to the ODE

$$-cU' + hu'' + UU' = 0$$

(25)

Integrating Eq. (25) gives

$$r - cU + huU' + \frac{1}{2}U^2 = 0,$$

(26)

where $r$ is the integral constant. Applying homogeneous balance to $U^2$ and $U'$ produces $n = 1$ for which the solution (4) becomes

$$U(\xi) = \frac{a_0 + a_1 (G'/G)}{b_0 + b_1 (G'/G)}$$

(27)

where at least one of $a_1$ or $b_1$ is non zero. Inserting Eq. (27) into Eq. (26) makes a polynomial in $(G'/G)$. Setting the coefficients to zero and calculating by computer software Maple provides the following values for $a_0, a_1, b_0, b_1$ and $c$:

$$a_0 = \frac{b_0}{2h} \left[ (c + h\lambda)(c + \sqrt{c^2 - 2r}) - 2r \right],$$

$$a_1 = b_1 (c + \sqrt{c^2 - 2r})$$

(28)

$$b_0 = \frac{b_0}{2h} \left[ h\lambda + \sqrt{c^2 - 2r} \right].$$

(29)

Eq. (27) together with the values in Eq. (28) reduces to

$$U(\xi) = \frac{\left( c + h\lambda \right)(c + \sqrt{c^2 - 2r}) - 2r + 2h(c + \sqrt{c^2 - 2r}) (G'/G)}{(h\lambda + \sqrt{c^2 - 2r}) + 2h(G'/G)}$$

where $\xi = x - ct$.

Eq. (29) with the aid of Eqs. (7)-(9) serves the following exact solutions to Eq. (24) in three types as hyperbolic, trigonometric and rational:

Case 1: If $\lambda^2 - 4\mu > 0$,

$$U_{14}(\xi) = \frac{\left( c + h\lambda \right)(c + \sqrt{c^2 - 2r}) - 2r + 2h(c + \sqrt{c^2 - 2r}) (G'/G)}{(h\lambda + \sqrt{c^2 - 2r}) + 2h(G'/G)}$$

(30)

where $\xi = x - ct$.

If $A = 0, B \neq 0$, then

$$U_{15}(\xi) = \frac{\left( c + h\lambda \right)(c + \sqrt{c^2 - 2r}) - 2r + 2h(c + \sqrt{c^2 - 2r}) (G'/G)}{(h\lambda + \sqrt{c^2 - 2r}) + 2h(G'/G)}$$

(31)

Case 2: For $\lambda^2 - 4\mu < 0$,

$$U_{16}(\xi) = \frac{\left( c + h\lambda \right)(c + \sqrt{c^2 - 2r}) - 2r + 2h(c + \sqrt{c^2 - 2r}) (G'/G)}{(h\lambda + \sqrt{c^2 - 2r}) + 2h(G'/G)}$$

(32)

Again, for $A \neq 0, B = 0$,

$$U_{17}(\xi) = \frac{\left( c + h\lambda \right)(c + \sqrt{c^2 - 2r}) - 2r + 2h(c + \sqrt{c^2 - 2r}) (G'/G)}{(h\lambda + \sqrt{c^2 - 2r}) + 2h(G'/G)}$$

(33)

where $\xi = x - ct$.

Assign $A = 0, B \neq 0$, then

$$U_{18}(\xi) = \frac{\left( c + h\lambda \right)(c + \sqrt{c^2 - 2r}) - 2r + 2h(c + \sqrt{c^2 - 2r}) (G'/G)}{(h\lambda + \sqrt{c^2 - 2r}) + 2h(G'/G)}$$

(34)

Fix $A \neq 0, B = 0$, then

$$U_{19}(\xi) = \frac{\left( c + h\lambda \right)(c + \sqrt{c^2 - 2r}) - 2r + 2h(c + \sqrt{c^2 - 2r}) (G'/G)}{(h\lambda + \sqrt{c^2 - 2r}) + 2h(G'/G)}$$

(35)

Case 3: When $\lambda^2 - 4\mu = 0$,

$$U_{20}(\xi) = \frac{\left( c + h\lambda \right)(c + \sqrt{c^2 - 2r}) - 2r + 2h(c + \sqrt{c^2 - 2r}) (G'/G)}{(h\lambda + \sqrt{c^2 - 2r}) + 2h(G'/G)}$$

(36)
where $\xi = x - ct$.

Put $A = 0, B \neq 0$,

$$u_n(\xi) = \frac{\{(c + h\lambda)(c \pm \sqrt{c^2 - 2r}) - 2r\} + 2h(c \pm \sqrt{c^2 - 2r})\left(-\frac{\lambda}{2} + \frac{1}{\xi}\right)}{(h\lambda \pm \sqrt{c^2 - 2r}) + 2h\left(-\frac{\lambda}{2} + \frac{1}{\xi}\right)}(37)$$

Choose $A \neq 0, B = 0$, then

$$u_n(\xi) = \frac{\{(c + h\lambda)(c \pm \sqrt{c^2 - 2r}) - 2r\} + 2h(c \pm \sqrt{c^2 - 2r})\left(-\frac{\lambda}{2}\right)}{(h\lambda \pm \sqrt{c^2 - 2r}) + 2h\left(-\frac{\lambda}{2}\right)}(38)$$

4 Results, discussion and graphical representations

To analyse the problems clearly and to describe the solutions of the phenomena, a graphical representation of the solutions can be an effective tool to brief the commentaries. On account of this, we provide different types of physical appearances of the solutions bearing the actual form of solitary waves (Fig. 1-Fig. 6). The plots are of kink shape soliton, cuspone, periodic solutions etc. Fig. 1 stands for kink type soliton of solution (17) for $\lambda = 3, \mu = 2, c = 1, a = 1, b = 1, x = 0$ in the interval $-10 \leq y \leq 10$ and $-10 \leq t \leq 10$. Fig. 2 characterizes cusp which is depicted for solution (17) for $\lambda = 3, \mu = 3, c = -1, a = 1, b = 1, x = 0$ within $-10 \leq y \leq 10$ and $-10 \leq t \leq 10$. Fig. 3 indicates periodic soliton generated from the solution (17) for $\lambda = 3, \mu = 3, c = 1, a = 1, b = 1, x = 0$ in $-10 \leq y \leq 10$ and $-10 \leq t \leq 10$. Fig. 4 indicates cusp plotted for solution (17) for $\lambda = 3, \mu = 2, c = -1, a = 1, b = 1, x = 0$ for the interval $-10 \leq y \leq 10$ and $-10 \leq t \leq 10$. Fig. 5 designates singular kink type soliton emerged from the solution (29) for $\lambda = 4, \mu = 2, c = -5, h = -5, r = 1$ within $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$. Fig. 6 represents kink soliton from the solution (29) for $\lambda = 7, \mu = 3, c = -5, h = -5, r = -1$ in the intervals $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$. Fig. 7 designates cusp emerged from solution (29) for $\lambda = 3, \mu = 4, c = 5, h = -1, r = -1$ for $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$. Fig. 8 designates singular kink type soliton emerged from the solution (29) for $\lambda = -3, \mu = 0, c = 5, h = -3, r = 1$ in the interval $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$. 

![Fig. 1](Image)

![Fig. 2](Image)

![Fig. 3](Image)

![Fig. 4](Image)

![Fig. 5](Image)
References


5 Conclusions

The determination of this article was to investigate closed form analytic solutions to the cKG and BO equations by employing the rational (G'/G)-expansion method. Consequently, different and advanced travelling wave solutions to the considered equations have successfully been furnished, on comparison to other methods available in the literature. Our derived solutions might effectively be helpful to depict the interior behaviors of internal mechanisms of nature world like describing shallow water waves, acoustic waves etc. The gained hyperbolic, trigonometric and rational function solutions together with the physical appearances show the efficiency and the reliability of our employed method which might be used in further research works to find fresh and further general solutions of any other NLPDEs in different fields.