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SVD-Krylov based Sparsity-preserving Techniques for Riccati-based Feedback Stabilization of Unstable Power System Models

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ABSTRACT

We propose an efficient sparsity-preserving reduced-order modelling approach for index-1 descriptor systems extracted from large-scale power system models through two-sided projection techniques. The projectors are configured by utilizing Gramian based singular value decomposition (SVD) and Krylov subspace-based reduced-order modelling. The left projector is attained from the observability Gramian of the system by the low-rank alternating direction implicit (LR-ADI) technique and the right projector is attained by the iterative rational Krylov algorithm (IRKA). The classical LR-ADI technique is not suitable for solving Riccati equations and it demands high computation time for convergence. Besides, in most of the cases, reduced-order models achieved by the basic IRKA are not stable and the Riccati equations connected to them have no finite solution. Moreover, the conventional LR-ADI and IRKA approaches do not preserve the sparse form of the index-1 descriptor systems, which is an essential requirement for feasible simulations. To overcome those drawbacks, the fitting of LR-ADI and IRKA based projectors from the left and right sides, respectively, desired reduced-order systems attained. So that, finite solution of low-rank Riccati equations, and corresponding feedback matrix can be executed. Using the mechanism of inverse projection, the Riccati-based optimal feedback matrix can be computed to stabilize the unstable power system models. The proposed approach will maintain minimized computation time and $\mathcal{H}_2$ -norm of the error system for reduced-order models of the target models.

Keywords: Singular Value Decomposition, Krylov Subspace, Alternative Direction Implicit, Riccati Equation, $\mathcal{H}_2$-norm, Optimal Feedback Stabilization.

1 Introduction

The index-1 descriptor systems of the first-order are conventionally the technical arrangement of sparse sub-matrices with appropriate structure. These sub-matrices are arranged in system-oriented input-output combinations. These systems can be formed as

$$\begin{aligned} E_{11}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \\ 0 &= A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t), \\ y(t) &= C_1x_1(t) + C_2x_2(t) + Du(t), \\ x(t_0) &= x_0, \quad t \geq t_0. \end{aligned} \tag{1}$$

In system (1), $E_{11} \in \mathbb{R}^{n_1\times n_1}$ is the differential coefficient matrix, and $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{12} \in \mathbb{R}^{n_1\times n_2}$, $A_{21} \in \mathbb{R}^{n_2\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$ are the state sub-matrices. The control multiplier sub-matrices $B_1 \in \mathbb{R}^{n_1\times p}$, $B_2 \in \mathbb{R}^{n_2\times p}$ with the state multiplier sub-matrices $C_1 \in \mathbb{R}^{m\times n_1}$, $C_2 \in \mathbb{R}^{m\times n_2}$. The direct gain matrix $D \in \mathbb{R}^{m\times p}$ is for the input to output transfer without alteration, it remains zero in many of the physical systems, for instance, power systems models. Here, the system dimension of the system (1) is $n = n_1 + n_2$ with the dimensions of input and output p and m, respectively. In the case of large-scale systems n is very large, whereas p, m are comparatively smaller. The vectors $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$ are for state vectors, whereas $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^m$ represent the input (control) and output vectors, respectively. The sub-matrices E_{11} and A_{22} have the full rank [1]-[2].

For further manipulation, $x_2(t) = -A_{22}^{-1}A_{21}x_1(t) - A_{22}^{-1}B_2u(t)$ needs to be eliminated from the algebraic (second) part of the Equation (1). Then the *Schur complements* of the system (1) can be formed as

$$\begin{aligned} x &:= x_1, \quad \mathcal{E} := E_{11}, \quad \mathcal{A} := A_{11} - A_{12}A_{22}^{-1}A_{21}, \\ \mathcal{B} &:= B_1 - A_{12}A_{22}^{-1}B_2, \mathcal{C} := C_1 - C_2A_{22}^{-1}A_{21}, \\ \mathcal{D} &:= D - C_2A_{22}^{-1}B_2. \end{aligned} \tag{2}$$

Using the Schur complements given in (2), the index-1 descriptor system (1) can structure into an analogous generalized LTI continuous-time system as

$$\begin{aligned} \mathcal{E}\dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t), \\ y(t) &= \mathcal{C}x(t) + \mathcal{D}u(t). \end{aligned} \tag{3}$$

Applying Laplace transformation, the transfer function for the system (3) can be found as

$$G(s) = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}; \quad s \in \mathbb{C}. \tag{4}$$

The implications of LTI continuous-time systems are inescapable in the branches of engineering fields with the applications of applied mathematics, for example, system and control theory, mechatronics, and power electronics [3]-[5].

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Continuous-time Algebraic Riccati Equation (CARE) plays a premier role in engineering applications, such as the systems that originated from mechanical and electrical fields [6]-[7] . The CARE yields from the system (3) can be formed as

$$\mathcal{A}^T X\mathcal{E} + \mathcal{E}^T X\mathcal{A} - \mathcal{E}^T X\mathcal{B}\mathcal{B}^T X\mathcal{E} + \mathcal{C}^T\mathcal{C} = 0. \qquad (5)$$

If all of the eigenvalues of the Hamiltonian matrix of the system (3) lie outside the imaginary axis, the solution X of the CARE (5) is then unique and finite [8]. The symmetric positive-definite Matrix X is called stabilizing if the closed-loop matrix $\mathcal{A} - (\mathcal{B}\mathcal{B}^T)X\mathcal{E}$ exists and is stable. Some of the systems, that have eigenvalues very close to the imaginary axis are called semi-stable systems. For an unstable type of system (3), the optimal feedback matrix $K^o = \mathcal{B}^T X\mathcal{E}$ needs to estimate to apply the Riccati-based feedback stabilization [9]. Implementing the desired K^o, the matrix $\mathcal{A}_s = \mathcal{A} - BK^o$ can be formed to replace the system matrix $\mathcal{A}$. Then the optimally stabilized target system can be written as

$$\begin{aligned} \mathcal{E}\dot{x}(t) &= \mathcal{A}_s x(t) + \mathcal{B}u(t), \\ y(t) &= \mathcal{C}x(t) + \mathcal{D}u(t). \end{aligned} \qquad (6)$$

In the previous works, we have discussed the rational Krylov subspace method (RKSM) and the Kleinman-Newton technique for the Riccati-based feedback stabilization technique for unstable index-1 descriptor systems [10]-[11]. The linear quadratic regulator (LQR) approach is the key to the RKSM and the low-rank Cholesky-factor integrated alternative direction implicit (LRCF-ADI) approach required in the Kleinman-Newton technique. A modified form of the iterative rational Krylov algorithm (IRKA) technique was recently derived to treat that of the systems through optimal feedback stabilization [12]. In those works, some unstable systems extracted from the Brazilian Interconnected Power System (BIPS) models are deliberated to stabilize.

In this work, we introduce two-sided projection techniques for Riccati-based feedback stabilization for unstable BIPS models utilizing reduced-order modeling, which is a coupled approach of singular value decomposition and Krylov subspace and naming as Iterative SVD-Krylov Algorithm (ISKA). The reduced-order models will be validated through the $\mathcal{H}_2$ norm of the error system. A comparative discussion on the present work and IRKA approaches will be done.

2 Preliminaries

The Lyapunov equation consisting of observability Gramian $\mathcal{Q}$ of the system (3) has the form

$$\mathcal{A}^T \mathcal{Q}\mathcal{E} + \mathcal{E}^T \mathcal{Q}\mathcal{A} + \mathcal{C}^T\mathcal{C} = 0. \qquad (7)$$

Computation of the $\mathcal{Q}$ by solving Equation (7) containing large-scale matrices by the direct solvers is infeasible for large-scale systems, sometimes it may be impossible for the rising size of the system components. Thus, the observability Gramian factor Z_q needs to be estimated by any feasible approach. There are some efficient techniques available for executing Z_q, for example, low-rank Cholesky-factor-based Alternating Direction Implicit (LRCF-ADI) [13]-[14]. Then the Gramian, $\mathcal{Q} = Z_q Z_q^T$ can be approximated as the solution of the Lyapunov Equation (7). The LRCF-ADI approach for computing Z_q is provided in Algorithm 1.

Algorithm 1: First-order LRCF-ADI Algorithm [13]

Input: $\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \tau$ (tolerance), i_{max} (iterations), and $\{\mu_j\}_{j=1}^{i_{max}}$ (initial shifts).

Output: Low-rank Cholesky-factor Z_q for $\mathcal{Q} = Z_q Z_q^T$.

```
Assume at i = 1, Z_0 = [ ] and W_0 = C^T.
while ||W_{i-1} W_{i-1}^T|| ≥ τ or i ≤ i_max do
    Solve V_i = (A^T + μ_i E^T)^{-1} W_{i-1}.
    if Im(μ_i) = 0 then
        Update Z_i = [Z_{i-1}   sqrt(-2μ_i) V_i],
        Compute W_i = W_{i-1} - 2μ_i E^T V_i.
    else
        Assume γ_i = sqrt(-2Re(μ_i)),   δ_i = Re(μ_i)/Im(μ_i),
        Update Z_{i+1} =
        [Z_{i-1}   γ_i(Re(V_i) + δ_i Im(V_i))   γ_i sqrt(δ_i^2 + 1) Im(V_i)],
        Compute, W_{i+1} = W_{i-1} - 4Re(μ_i) E^T [Re(V_i) + δ_i Im(V_i)].
        i = i + 1
    end if
    i = i + 1
end while
```

Considering the computationally feasible r-dimensional reduced-order model (ROM) of the system (3) as

$$\begin{aligned} \hat{\mathcal{E}}\dot{\hat{x}}(t) &= \hat{\mathcal{A}}\hat{x}(t) + \hat{\mathcal{B}}\hat{u}(t), \\ \hat{y}(t) &= \hat{\mathcal{C}}\hat{x}(t) + \hat{\mathcal{D}}\hat{u}(t), \end{aligned} \qquad (8)$$

where $\hat{\mathcal{E}} \in \mathbb{R}^{r\times r}$, $\hat{\mathcal{A}} \in \mathbb{R}^{r\times r}$, $\hat{\mathcal{B}} \in \mathbb{R}^{r\times p}$, $\hat{\mathcal{C}} \in \mathbb{R}^{m\times r}$ and $\hat{\mathcal{D}} \in \mathbb{R}^{m\times p}$.

The reduced coefficient matrices of (8) are formed by the following way

$$\hat{\mathcal{E}} = W^T\mathcal{E}V, \quad \hat{\mathcal{A}} = W^T\mathcal{A}V, \quad \hat{\mathcal{B}} = W^T\mathcal{B}, \quad \hat{\mathcal{C}} = \mathcal{C}V, \quad \hat{\mathcal{D}} = \mathcal{D}. \qquad (9)$$

The transfer function of the ROM (8) can be found as

$$\hat{G}(s) = \hat{\mathcal{C}}\left(s\hat{\mathcal{E}} - \hat{\mathcal{A}}\right)^{-1}\hat{\mathcal{B}} + \hat{\mathcal{D}}; \quad s \in \mathbb{C}. \qquad (10)$$

The right projector V is built by the well-known Krylov-based interpolatory techniques IRKA given in [15]-[16] as

$$V = [(\alpha_1\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}b_1, \dots, (\alpha_r\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}b_r], \qquad (11)$$

where $\{\alpha_i\}_{i=1}^r$ and $\{b_i\}_{i=1}^r$ are the interpolation points and tangential direction respectively. The left projector W is computed by the observability Gramian $\mathcal{Q}$ utilizing the singular value decomposition-based techniques discussed in [17]-[19] as

$$W = \mathcal{Q}V(V^T\mathcal{Q}V)^{-1}. \qquad (12)$$

The successive steps of the computation of the ROM (8) are exhibited in Algorithm 2.

Algorithm 2: First-order ISKA [17]

Input: $\mathcal{E}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and Z_q (from Algorithm 1).

Output: $\hat{\mathcal{E}}, \hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}} \coloneqq \mathcal{D}$.

1 Choose the initial interpolation points $\{\alpha_i\}_{i=1}^{r}$ and the tangential directions $\{b_i\}_{i=1}^{r}$.
2 Construct $V = [(\alpha_1\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}b_1, \dots, (\alpha_r\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}b_r]$.
3 Compute $\mathcal{Q} = Z_q Z_q^T$ and construct $W = \mathcal{Q}V(V^T\mathcal{Q}V)^{-1}$.
4 **while** $(not\ converged)$ **do**
5 Find $\hat{\mathcal{E}} = W^T\mathcal{E}V,\ \hat{\mathcal{A}} = W^T\mathcal{A}V,\ \hat{\mathcal{B}} = W^T\mathcal{B},\ \hat{\mathcal{C}} = \mathcal{C}V$.
6 **for** $i = 1, \dots, r$. **do**
7 Evaluate $\hat{\mathcal{A}}z_i = \lambda_i\hat{\mathcal{E}}z_i$ and $y_i^*\hat{\mathcal{A}} = \lambda_i y_i^*\hat{\mathcal{E}}$ to find $\alpha_i \leftarrow -\lambda_i, b_i^* \leftarrow -y_i^*\hat{\mathcal{B}}$.
8 **end for**
9 Repeat Step-2 and Step-3.
10 $i = i + 1$
11 **end while**
12 Repeat Step-5 to find the reduced-order matrices.

3 Sparsity-preserving SVD-Krylov techniques for the stabilization of first-order index-1 descriptor system

The objective of this work is to reduce the dimension of the first-order index-1 system (1) by keeping the sparse structure invariant through the Iterative SVD-Krylov Algorithm (ISKA) approach. To do this, it is essential to modify some steps of first-order ISKA and LRCF-ADI algorithms in terms of sparse sub-matrices.

3.1 Sparsity-preserving LRCF-ADI approach for the first-order index-1 descriptor system

The LRCF-ADI method of first-order was discussed in [13],[20]-[21]. The modification of the LRCF-ADI algorithm for the structure-preserving first-order index-1 descriptor form can be derived as follows.

For the truncated term Γ, the first iteration of the Step-3 of Algorithm 1 can be written as

$$(\mathcal{A}^T + \mu_1\mathcal{E}^T)\mathcal{V}_1 = \mathcal{C}^T,$$
$$or, \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T + \mu_1 \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}^T\right)\begin{bmatrix} \mathcal{V}_1 \\ \Gamma \end{bmatrix} = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix}. \tag{13}$$

Thus, we have

$$\begin{bmatrix} A_{11}^T + \mu_1 E_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}\begin{bmatrix} \mathcal{V}_1 \\ \Gamma \end{bmatrix} = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix}. \tag{14}$$

As a consequence, for $i \geq 2$, the next $i - th$ iteration takes the form

$$\begin{bmatrix} A_{11}^T + \mu_i E_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}\begin{bmatrix} \mathcal{V}_i \\ \Gamma \end{bmatrix} = \begin{bmatrix} \mathcal{W}_{i-1} \\ 0 \end{bmatrix}. \tag{15}$$

If the estimated shift parameter has no imaginary part, then the Step-6 of Algorithm 1 can be formed as

$$\mathcal{W}_i = \mathcal{W}_{i-1} - 2\mu_i E_{11}^T \mathcal{V}_i. \tag{16}$$

Otherwise, for $\delta_i = \frac{Re(\mu_i)}{Im(\mu_i)}$, the Step-10 of Algorithm 1 can be expressed as

$$\mathcal{W}_{i+1} = \mathcal{W}_{i-1} - 4Re(\mu_i)E_{11}^T[Re(\mathcal{V}_i) + \delta_i Im(\mathcal{V}_i)]. \tag{17}$$

The restructured sparse form of LRCF-ADI for the first-order index-1 descriptor system is exhibited in Algorithm 3.

Algorithm 3: LRCF-ADI Algorithm for first-order sparse index-1 descriptor system

Input: $E_{11}, A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, \tau$ (tolerance), i_{max} (iterations), and $\{\mu_j\}_{j=1}^{i_{max}}$ (initial shifts).

Output: Low-rank Cholesky-factor Z_q for $\mathcal{Q} \approx Z_q Z_q^T$.

1 Assume at $i = 1, Z_0 = [\,]$ and $\mathcal{W}_0 = [C_1 \quad C_2]^T$.
2 **while** $\|\mathcal{W}_{i-1}\mathcal{W}_{i-1}^T\| \geq \tau$ or $i \leq i_{max}$ **do**
3 Solve (14) to find $\mathcal{V}_1$ and (15) to find $\mathcal{V}_i$; $i \geq 2$.
4 **if** $Im(\mu_i) = 0$ **then**
5 Update $Z_i = [Z_{i-1} \quad \sqrt{-2\mu_i}\mathcal{V}_i]$,
6 Compute the updated value of $\mathcal{W}_i$ by (16)
7 **else**
8 Assume $\gamma_i = \sqrt{-2Re(\mu_i)}$, $\delta_i = \frac{Re(\mu_i)}{Im(\mu_i)}$,
9 Update $Z_{i+1} = \left[Z_{i-1} \quad \gamma_i(Re(\mathcal{V}_i) + \delta_i Im(\mathcal{V}_i) \quad \gamma_i\sqrt{\delta_i^2 + 1}Im(\mathcal{V}_i)\right]$,
10 Compute the updated value of $\mathcal{W}_{i+1}$ by (17).
11 $i = i + 1$
12 **end if**
13 $i = i + 1$
14 **end while**

3.2 Sparsity-preserving ISKA for first-order index-1 descriptor system

Algorithm 2 needs to reform in the sparse form with the system matrices of (1). In the Step-2 of this algorithm, projector V needs to be re-structured utilizing the first-order sparse matrices. Let us consider the $i - th$ iteration of V be expressed as V_i and it can be configured as

$$(\alpha_i\mathcal{E} - \mathcal{A})V_i = \mathcal{B}b_i,$$
$$or, \begin{bmatrix} \alpha_i E_{11} - A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix}\begin{bmatrix} V_i \\ \Lambda \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} b_i. \tag{18}$$

The term Λ is to be truncated. The explicit execution of the reduced-order matrices for the system (8) defined in (9) is infeasible and contradicts the aim of the work.

The reduced-order matrices can be efficiently acquired by the sparsity-preserving form as

$$\begin{aligned} \hat{\mathcal{E}} &= W^T E_{11} V, \quad \hat{A} = W^T A_{11} V - (W^T A_{12})A_{22}^{-1}(A_{21}V), \\ \hat{\mathcal{B}} &= W^T B_1 - (W^T A_{12})A_{22}^{-1}B_2, \\ \hat{\mathcal{C}} &= C_1 V - C_2 A_{22}^{-1}(A_{21}V). \end{aligned} \tag{19}$$

The sparsity-preserving modified form of Algorithm 2 for the first-order index-1 descriptor system is summarized in Algorithm 4.

Algorithm 4: ISKA for first-order sparse index-1 descriptor system

Input: $E_{11}, A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D$ and Z_q (from Algorithm 3).
Output: $\hat{\mathcal{E}}, \hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}}, \hat{\mathcal{D}} \coloneqq D - C_2 A_{22}^{-1} B_2$.

1 Choose the initial interpolation points $\{\alpha_i\}_{i=1}^{r}$ and the tangential directions $\{b_i\}_{i=1}^{r}$.
2 Construct $V = [V_1, V_2, \dots, V_r]$ using (18).
3 Compute $Q = Z_q Z_q^T$ and construct $W = QV(V^T QV)^{-1}$.
4 **while** ($not\ converged$) **do**
5 Find the reduced-order matrices by (19).
6 **for** $i = 1, \dots, r$. **do**
7 Evaluate $\hat{\mathcal{A}} z_i = \lambda_i \hat{\mathcal{E}} z_i$ and $y_i^* \hat{\mathcal{A}} = \lambda_i y_i^* \hat{\mathcal{E}}$ to find $\alpha_i \longleftarrow -\lambda_i, b_i^* \longleftarrow -y_i^* \hat{B}$.
8 **end for**
9 Repeat Step-2 and Step-3.
10 $i = i + 1$
11 **end while**
12 Repeat Step-5 to find the reduced-order matrices.

3.3 Computing the optimal feedback matrix from ROM

System (3) can be written in a reduced-order form as (8) by exerting the reduced-order matrices defined in (19) and corresponding CARE can be attained as

$$\hat{\mathcal{A}}^T \hat{X} \hat{\mathcal{E}} + \hat{\mathcal{E}} \hat{X} \hat{\mathcal{A}} - \hat{\mathcal{E}} \hat{X} \hat{\mathcal{B}} \hat{\mathcal{B}}^T \hat{X} \hat{\mathcal{E}} + \hat{\mathcal{C}}^T \hat{\mathcal{C}} = 0. \tag{20}$$

Here $\hat{X}$ has the same properties of X. The MATLAB library command *care* can be applied to solve the low-rank CARE (20). The low-rank stabilizing feedback matrix $\hat{K} = \hat{\mathcal{B}}^T \hat{X} \hat{\mathcal{E}}$ corresponding to the ROM (8) can be computed and consequently approximated the stabilizing optimal feedback matrix K^o of the full model (3) can be reclaimed employing the scheme of reverse projection as

$$K^o = (\hat{\mathcal{B}}^T \hat{X} \hat{\mathcal{E}}) V^T E_{11} = \hat{K} V^T E_{11}. \tag{21}$$

3.4 Optimally stabilized first-order index-1 descriptor system

For the original system (1), optimal feedback matrix K^o can be attained by assigning the ROM (8). Then by utilizing K^o, the optimally stabilized system (1) can be found by replacing A_{11} and A_{21} by $A_{11} - B_1 K^o$ and $A_{21} - B_2 K^o$, respectively

3.5 $\boldsymbol{\mathcal{H}_2}$-norm of the error system

Now, the error system associated with the ROM (8) of the subjected system (1) by maintaining the form (3) has the form

$$G_{err} = G(s) - \hat{G}(s) = \mathcal{C}_{err}(s\mathcal{E}_{err} - \mathcal{A}_{err})^{-1} B_{err}, \tag{22}$$

where the transfer functions $G(s)$ and $\hat{G}(s)$ are connected to the systems (1) and (8), respectively. In (22), we have constituted

$$\mathcal{E}_{err} = \begin{bmatrix} \mathcal{E} & 0 \\ 0 & \hat{\mathcal{E}} \end{bmatrix}, \quad \mathcal{A}_{err} = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \hat{\mathcal{A}} \end{bmatrix}, \\ \mathcal{B}_{err} = \begin{bmatrix} \mathcal{B} \\ \hat{\mathcal{B}} \end{bmatrix} \quad and \quad \mathcal{C}_{err} = [\mathcal{C} \quad -\hat{\mathcal{C}}]. \tag{23}$$

The observability Lyapunov equation corresponding to the Graminan Q_{err} of the error system (22) is

$$\mathcal{A}_{err}^T Q_{err} \mathcal{E}_{err} + \mathcal{E}_{err}^T Q_{err} \mathcal{A}_{err} + \mathcal{C}_{err}^T \mathcal{C}_{err} = 0. \tag{24}$$

For the error system (22), the Authors in [5] explored an efficient approach to approximate the $\mathcal{H}_2$-norm as

$$\|G_{err}\|_{\mathcal{H}_2} = \sqrt{trace(\mathcal{B}_{err}^T Q_{err} \mathcal{B}_{err})} \\ = \sqrt{\|G(s)\|_{\mathcal{H}_2}^2 + \|\hat{G}(s)\|_{\mathcal{H}_2}^2 + 2trace(\mathcal{B}^T Q_s \hat{\mathcal{B}})} \tag{25}$$

Here, $\|G(s)\|_{\mathcal{H}_2}$ is the $\mathcal{H}_2$-norm of the full model which we need to evaluate at one time in computation but this is unfeasible to investigate for a large-scale system by any direct solver. Suppose Z_q is the low-rank Gramian factor that can be successfully determined by rearranging Algorithm 3, such that $Q = Z_q Z_q^T$, then $\|G(s)\|_{\mathcal{H}_2}$ can be written as

$$\|G(s)\|_{\mathcal{H}_2}^2 = trace(\mathcal{B}^T Q \mathcal{B}) \\ = trace\left(B_1^T (Z_q Z_q^T) B_1 + B_2^T (Z_q Z_q^T) B_2\right). \tag{26}$$

Again, the $\mathcal{H}_2$-norm of the ROM, $\|\hat{G}(s)\|_{\mathcal{H}_2}$ can be enumerated by the Gramian $\hat{Q}$ of the low-rank Lyapunov equation

$$\hat{\mathcal{A}}^T \hat{Q} \hat{\mathcal{E}} + \hat{\mathcal{E}}^T \hat{Q} \hat{\mathcal{A}} + \hat{\mathcal{C}}^T \hat{\mathcal{C}} = 0, \tag{27}$$

that consists of reduced-order matrices. Due to the small size of these matrices, the following Lyapunov equation is solvable by the MATLAB library command *lyap*.

Finally, $trace(\mathcal{B}^T Q_s \hat{\mathcal{B}})$ can be measured by the low-rank Gramian Q_s of the sparse-dense Sylvester equation

$$\mathcal{A}^T Q_s \hat{\mathcal{E}} + \mathcal{E}^T Q_s \hat{\mathcal{A}} + \mathcal{C}^T \hat{\mathcal{C}} = 0, \tag{28}$$

that can be efficiently solved by the techniques presented in Algorithm 4 of [22].

4 Numerical results

The derived method ISKA is validated by implementing some models evolved from Brazilian Interconnected Power System (BIPS) [23]. The computations are performed numerically by MATLAB® R2015a (8.5.0.197613) with a processor $4 \times$ Intel®Core™i5 $-$ 6200U incorporating a memory capacity of 16 GB with a clock speed of 2.30 GHz.

Table 1 displays the dimensions of the discussed models along with analogous input-output structures, and the size of the corresponding ROMs gained by the developed technique ISKA as illustrated in Algorithm 4.

Table 1 Model examples with input-output structures

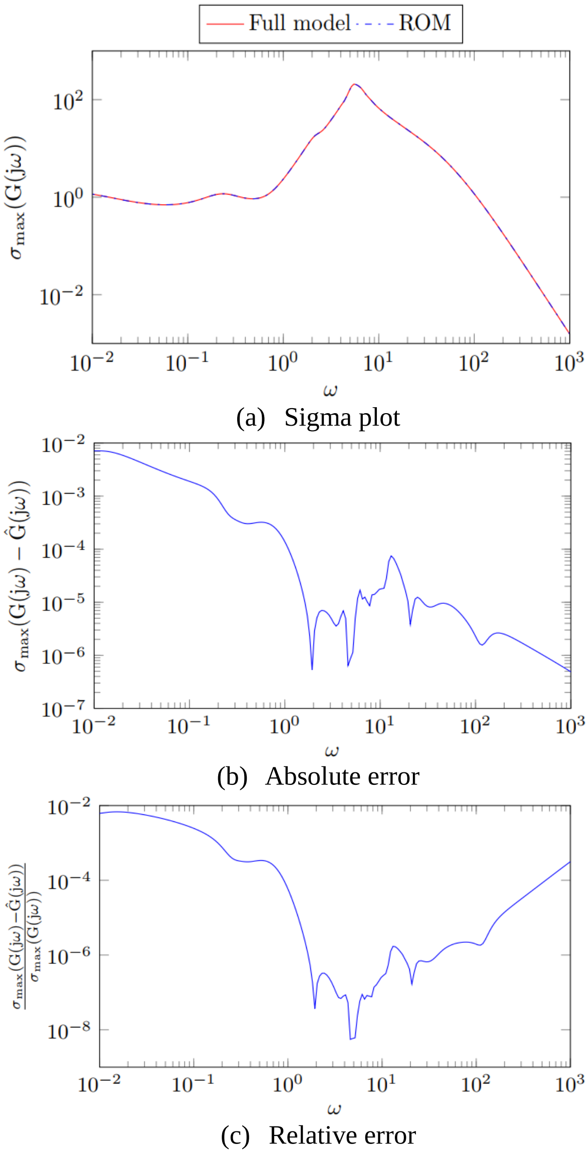
Model	Full model (n)	Input/Output	ROM (r)
BIPS-606	7135	4/4	30
BIPS-1998	15066	4/4	70
BIPS-2476	16861	4/4	100
BIPS-3078	21128	4/4	120

The models are named on the number of states they consist of [24]. Detailed of those models are available on the web-page.

In the previous work [12], it is found that IRKA based reduced-order model for the semi-stable model BIPS-3078 has no finite solution of the corresponding Riccati equation but the ISKA approach has no such kind of limitation.

4.1 Frequency domain analysis

All the above-mentioned power system models are structurally identical and have the same physical attributes. For the compactness of the work, only the comparative analysis of the transfer function of model BIPS-3078 is provided.



(a) Sigma plot

(b) Absolute error

(c) Relative error

Fig. 1 Comparison of the full model and ROM of the model BIPS-3078

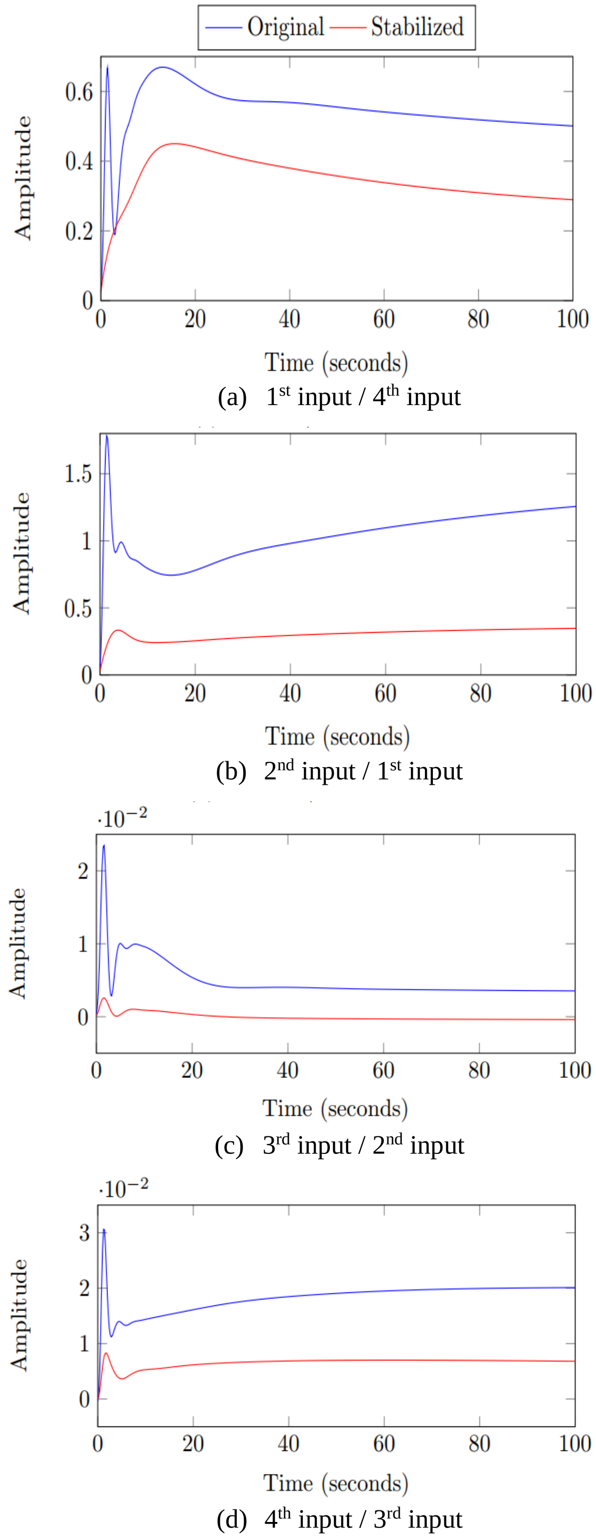
Sub-figures of Fig. 1 give graphical validation of the efficient match of the transfer function of model BIPS-3078 with the corresponding ISKA-based ROM. Fig. 1a displays comparison transfer functions, whereas Fig. 1b and Fig. 1c depict the absolute error and relative error in computing the ROM of the target model.

From the above-displaying figures, it can be said that the ISKA-based approach is efficient for finding desired ROM of the target model.

4.2 Stability analysis

Sub-figures of Fig. 2 exhibit the stabilized step-responses of the model BIPS-3078 for the dominant input-output relations.

From the foregoing figures, it has been seen that in every input-output relation, the step-responses of the model BIPS-3078 optimally stabilized.



(a) 1st input / 4th input

(b) 2nd input / 1st input

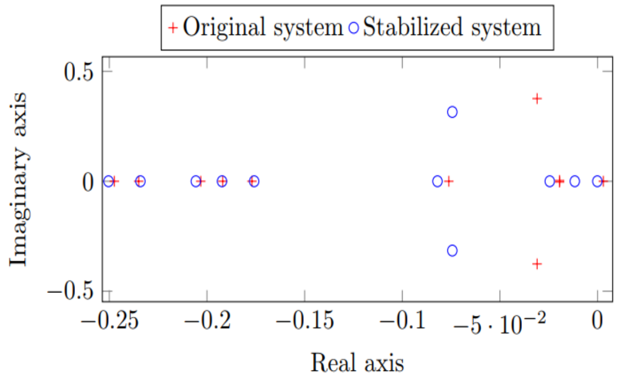
(c) 3rd input / 2nd input

(d) 4th input / 3rd input

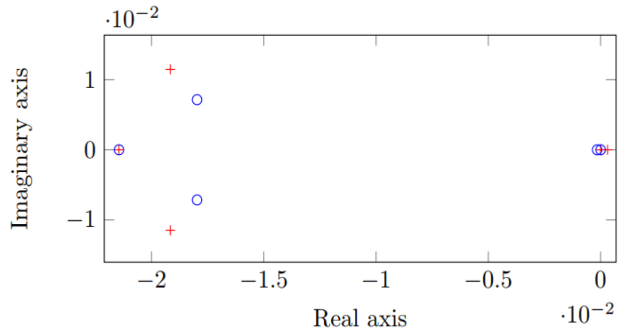
Fig. 2 Stabilization of step-responses of the model BIPS-3078

Sub-figures of Fig. 3 demonstrate the optimal feedback stabilization of the eigenvalues of the BIPS models.

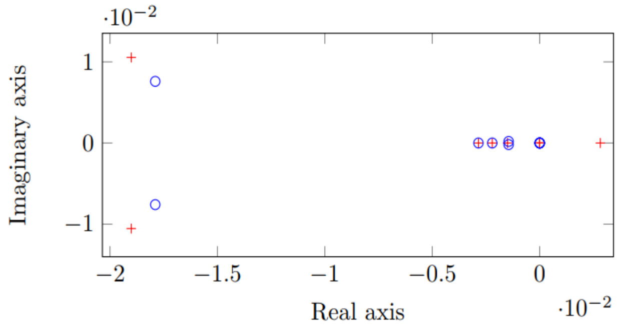
From the aforesaid figures, it is revealed that the eigenvalues of the models BIPS-606, BIPS-1998, and BIPS-2476 are stabilized efficiently.



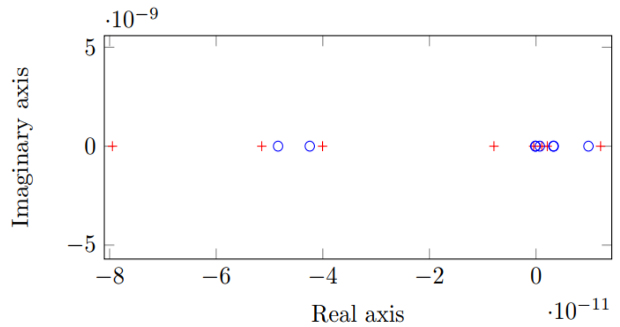
(a) BIPS-606



(b) BIPS-1998



(c) BIPS-2476



(d) BIPS-3078

Fig. 3 Eigenvalue stabilization of BIPS models

But the stabilization process for the semi-stable model BIPS-3078 is mildly interrupted but still acceptable as the eigenvalues are in the very small neighborhood of the imaginary axis (a very magnified view is provided).

4.3 $\mathcal{H}_2$-norm comparisons

Table 2 represents the $\mathcal{H}_2$ error norm of the full model and the corresponding ROM of the BIPS models by the ISKA and IRKA approaches. This table is numerical evidence of the robustness of the proposed approach.

Table 2 $\mathcal{H}_2$ error norm of the full models and the ROMs

Model	BIPS-606	BIPS-1998	BIPS-2476	BIPS-3078
ISKA	2.16×10^{0}	1.22×10^{-2}	4.34×10^{-4}	2.05×10^{-4}
IRKA	1.95×10^{0}	4.31×10^{-3}	1.24×10^{-3}	9.01×10^{-4}

It has been audited that the $\mathcal{H}_2$ error norms are decreasing with the increasing size of the target models and ISKA approach outplayed the IRKA approach. Thus, the ISKA approach is expedient to minimize the $\mathcal{H}_2$ error norms of the considering models and it gives a better approximation for the models of the larger size.

Table 3 depicts the computation time for ROM of the BIPS models through the ISKA and IRKA approaches.

Table 3 Computation time for the ROMs

Model	BIPS-606	BIPS-1998	BIPS-2476	BIPS-3078
ISKA	2.78×10^{2}	1.12×10^{3}	1.19×10^{3}	1.25×10^{3}
IRKA	4.11×10^{2}	4.89×10^{3}	4.92×10^{3}	5.17×10^{3}

From the tabular comparison, it has been perceived that for all of the BIPS models, the computation time for ROM in the proposed ISKA approach is significantly preferable to the IRKA approach.

5 Conclusion

We have discussed a sparsity-preserving two-sided projection-based reduced-order modelling approach for index-1 descriptor systems of the first-order form. The Gramian based singular value decomposition and Krylov-based reduced-order modelling are coupled to achieve the reduced-order models. The conventional first-order LRCF-ADI and ISKA algorithms are modified to the sparse form keeping the structure of the target systems invariant. The proposed techniques are devoted to the Riccati-based feedback stabilization of the target systems. We have tested the validity of the proficiency of the derived approaches by implementing them to the power system models of the type unstable index-1 descriptor system in the first-order form. The robustness and feasibility are investigated by the computational time and the $\mathcal{H}_2$ error norms of the reduced-order models corresponding to the target models.

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